# Wall shear stress in parallel plate flow chamber as a function of flow rate

EBS 289F
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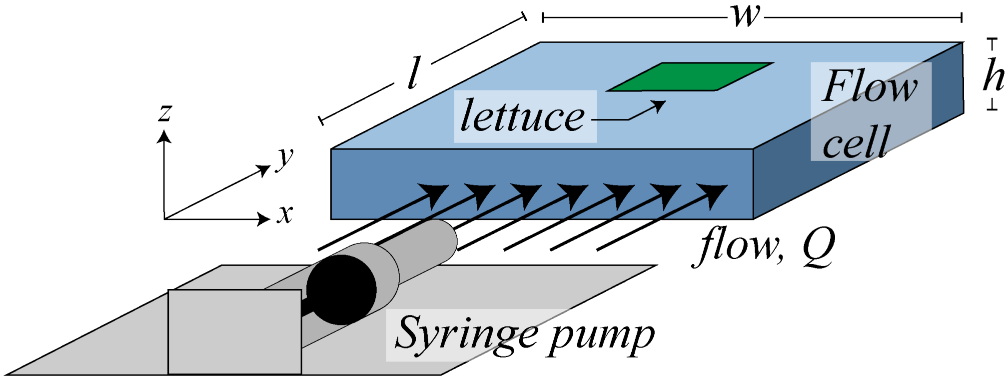
## 1 Motivation

In this project we propose to place a piece of lettuce on the wall of a parallel plate flow chamber and subject it to controlled, oscillating shear stress conditions. What we want to be able to do is calculate the shear stress on a section of wall in terms of a process variable that can be controlled in the laboratory. Since syringe pumps can deliver precise flow rates both in forward and in reverse operation a good option would be to connect a syringe pump to the flow cell, and then have it run a program of known flow rates varying back and forth from $Q = -Q_1$ to $Q = +Q_1$. We need a way to relate shear stress on the wall, $\tau_{wall}$, to the flow rate through the chamber, $Q$.




## 2 Informal derivation

First consider a three dimensional, rectangular prism with fluid flowing through it. The volumetric flow rate is given by

$$Q = \int_0^w \int_0^h v_y(x,z)dxdz$$

By arguing that the chamber is much wider than it is tall ($w >> h$), we can say that the velocity profile varies only in the vertical direction, thus $v_y(x,z) \approx v_y(z)$. Now we have

$$Q = w \int_0^h v_y(z)dz$$

What we want to do here is figure out how to get $V_y(z)$ as a function of $Q$, a known volumetric flow rate. To do this, we need to figure out what kind of shape the flow profile should have. We can come up with an answer by considering the conservation of momentum equation for a newtonian, incompressible fluid, which is the well known Navier-Stokes equation

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) = \rho \vec{g} - \vec{\nabla} P + \mu \nabla^2 \vec{v}$$

Where vector quantities have arrows. Since we are only interested in the flow through the device, we can take just the $y$ component of this equation. Since gravity is in the $-z$ direction, $g_y = 0$. In the case of a steady state flow, $\frac{\partial \vec{v}}{\partial t} = 0$. Finally, $\vec{v} \cdot \vec{\nabla} \vec{v} = 0$ for unidirectional or incompressible flows. Ours is both unidirectional and incompressible. All we have left is

$$0 = -\frac{1}{\rho} \vec{\nabla} P + \frac{\mu}{\rho} \nabla^2 \vec{v}$$

But we only want the $y$ component, so this is actually

$$0 = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \frac{\mu}{\rho} \frac{\partial^2}{\partial z^2} v_y$$

Integrating once with respect to $z$ gives us

$$C_1 = -\frac{1}{\rho} \frac{\partial P}{\partial y} z + \frac{\mu}{\rho} \frac{\partial}{\partial z} v_y$$

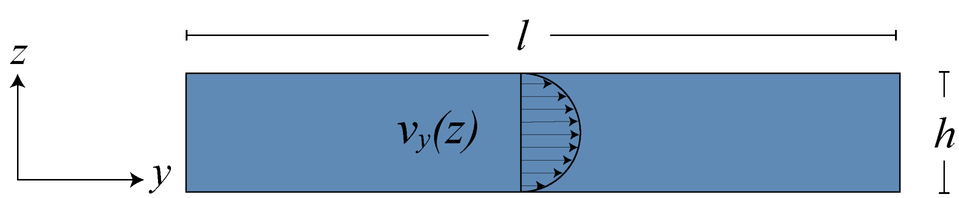
Integrating again with respect to $z$ gives

$$C_1 z + C_2 = -\frac{1}{\rho} \frac{\partial P}{\partial y} z^2 + \frac{\mu}{\rho} v_y$$

The important thing is that we have found that for any pressure drop in the direction of flow, $\frac{\partial P}{\partial y}$, the flow profile has some kind of parabolic shape in the $z$ direction. Another way of writing this is that

$$v_y(z) = Az^2 + Bz + C$$

This can be visualized with a side view of the flow cell.




Now, we can use a no-slip boundary condition at $z = 0$, the bottom of the chamber. This is nothing but saying that $v_y = 0$ at $z = 0$, which tells us that $C = 0$. We can also use a no-slip boundary condition at the top of the chamber, which means that

$$v_y\big|_{z=h} = A(h)^2 + B(h) = 0$$

Thus

$$B = -Ah$$

And the velocity is now given by

$$v_y(z) = Az^2 - Ahz$$

Returning to the second equation that was given, and substituting for the previously unknown velocity profile

$$Q = w \int_0^h (Az^2 - Ahz)dz$$

We can solve this because the flow rate, $Q$, is something set experimentally. It is known, Khaleesi. The only unknown is the constant, $A$. So we integrate and get

$$Q = wA \left[ \frac{z^3}{3} - \frac{hz^2}{2} \right] \Bigg|_{z=0}^{z=h}$$

$$Q = -wAh^3/6$$

$$A = -6Q/wh^3$$

Now we can go back to the flow profile and put it in terms of the flow rate.

$$v_y(z) = (-6Q/wh^3)z^2 - (-6Q/wh^3)hz$$

Lastly, for a newtonian fluid in a unidirectional flow such as this one, the shear stress is given by

$$\tau = \mu \frac{\partial v_y}{\partial z}$$

Since we know the velocity profile, we can substitute and find that

$$\tau = \mu \frac{\partial}{\partial z} \left( \frac{-6Q}{wh^3} z^2 + \frac{6Q}{wh^2} z \right)$$

Now we evaluate this at the top wall (where we will put the lettuce) and get

$$\tau = \mu \left( \frac{-12Q}{wh^3} z + \frac{6Q}{wh^2} \right) \Bigg|_{z=h}$$

$$\boxed{\tau = \frac{-6\mu Q}{wh^2}}$$

Where:
$\tau$ = shear stress acting on the wall ($Pa$)
$\mu$ = viscosity of the fluid ($Pa - s$)
$w$ = width of the flow chamber ($m$)
$h$ = height of the flow chamber ($m$)
$Q$ = volumetric flow rate ($m^3/s$)

The stress will be defined as negative for forward flows, and positive when we reverse the flow through the chamber (see the introduction). Note that Wikipedia offers a different equation, but some authors have come up with an equation that agrees with mine [1]. Neither offer a derivation though.

## 3 Recap

We have found an equation to relate a known flow through the parallel plate cell with a shear stress on the lettuce. These parallel plate devices typically have gap height, $h$, of roughly $100 - 200\mu m$, width, $w$, of roughly $2 - 3cm$, and length, $l$, between $5 - 7cm$. The shear stress on the wall can be varied between $0.5 - 5Pa$ by changing the flow rate, but it should be noted that the Reynolds number, given by $\rho Q/\mu w$, must be less than about 100 to ensure that the flow remains laminar [1].

## References


[1] Luigi Preziosi. *Cancer modeling and simulation.* Chapman and Hall/CRC ISBN 1-58488-361-8, pg 255, 2003.